M_k is k-th Hirzebruch surface

M_k is $\mathbb{P}(\mathcal{O}(k) \oplus \mathcal{O})$ over $\mathbb{P}^1$.

D_∞ is section given by $(0,1)$ } switched from usual

D_0 is section given by $(1,0)$

Topology gives that $[D_\infty]$ & $[\mathcal{C}]$ span $H^{1,1}$.

Also : $\circ$ $\mathcal{C}^2 = 0$ $\qquad D_\infty^2 = k$ ②

$\circ$ $\mathcal{C} \cdot D_\infty = 1$

$\circ$ $\mathcal{C} \cdot D_0 = 1$ ①

$\circ$ $D_0 = m D_\infty + l\,\mathcal{C}$

① $\Rightarrow$ $m = 1$

② $\Rightarrow$ $l = -k$

$\Big\} \Rightarrow D_0 = D_\infty - k\mathcal{C}$

$\circ$ $D_0^2 = (D_\infty - k\mathcal{C})^2 = -k$

Since D_∞ & $\mathcal{C}$ span $H^{1,1}$ a Kähler form is:

$$
\begin{aligned}
[\omega] &= \alpha[D_\infty] + \beta[\mathcal{C}] \\
&= \alpha[D_\infty] + \frac{\beta}{k}\left([D_\infty] - [D_0]\right) \\
&= \frac{1}{k}\left[(k\alpha + \beta)[D_\infty] - \beta[D_0]\right)
\end{aligned}
$$

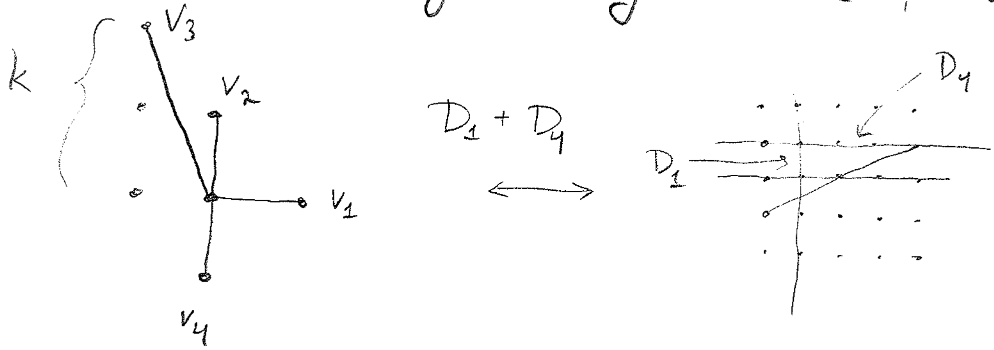
So $\quad [\omega] = \frac{b}{k}[D_\infty] - \frac{a}{k}[D_0],$

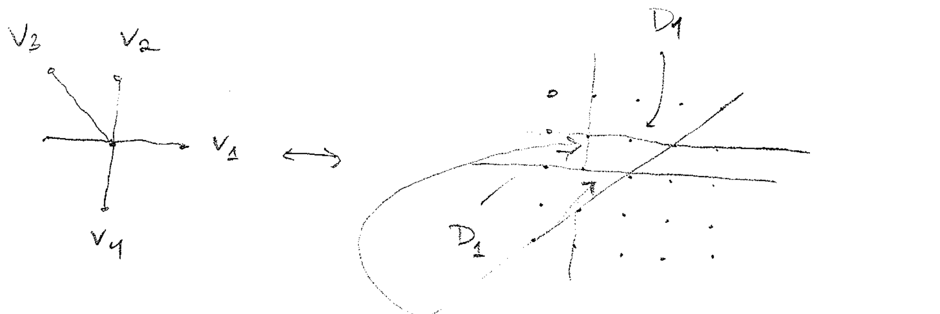
w/ $b > a > 0$. Specifically

- $b = (k\alpha + \beta)$
- $a = \beta$

(stupid notation on my part)

Fan for M_k given by $\quad$ (D_1 & D_4 span)

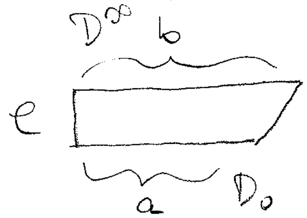




If $\quad \alpha[D_0] + \beta[\mathcal{C}]$ then this is α

and this is β so top piece is $\beta + k\alpha$.

So point is (for M_1), polytope $\leftrightarrow$ metric give by



$\longleftrightarrow \quad \frac{b}{k}[D_\infty] - \frac{a}{k}[D_0]$

Note e_1 is topological!

Computation gives:

$$K_M^{-1} = 2[D_\infty] - (k-2)[e]$$

$$= 2[D_\infty] - \frac{(k-2)}{k}\left([D_\infty] - [D_0]\right)$$

$$= \frac{2k - k + 2}{k}[D_\infty] + \frac{k-2}{k}[D_0]$$

$$= \frac{k+2}{k}[D_\infty] + \frac{k-2}{k}[D_0]$$

Under Kähler-Ricci flow,

$$[\mathring{\omega}] = -K_M^{-1}$$

$$= -\frac{k+2}{k}[D_\infty] - \frac{k-2}{k}[D_0]$$

So $\quad [\omega] = \frac{b}{k}[D_\infty] - \frac{a}{k}[D_0]$

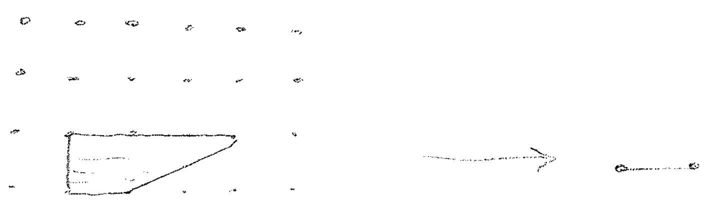
Means

- $b_t = b_0 - t(k+2)$
- $a_t = a_0 + t(k-2)$

So if $k \geq 2$, a_t is increasing!

But b_t is *always* decreasing!

So if $k \geq 2$ have picture:



(length is a_T!)

So seems like fibers collapse and become $\mathbb{P}^1$!

This is true!

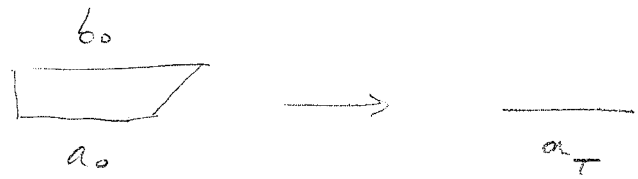
Now if $k = 1$, both a_t & b_t are decreasing!

It's a race!

- $b_t = b_0 - 3t$
- $a_t = a_0 - t$

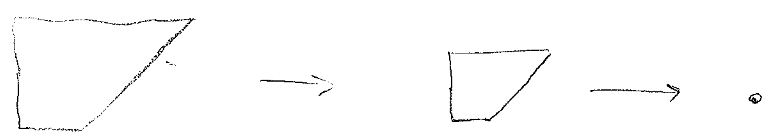
Now 3 cases

- $b_0 < 3a_0 \implies b_t \to 0$ first!



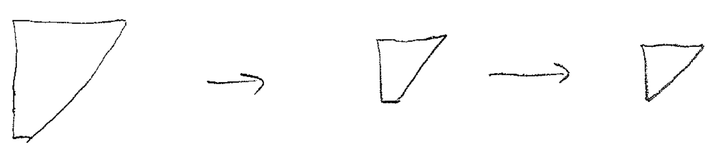
Same as before!

- $b_0 = 3a_0$



So becomes a pt!

- $b_0 > 3a_0$



So becomes $\mathbb{P}^2$. Contracts exceptional divisor!

Want to study flow under Calabi symmetry.

First give parametrization of M_k:

$$(x_1, x_2) \mapsto ([x_1, x_2], \text{ something in fiber})$$

First $\mathbb{P}^1 \ni [x_1, x_2]$ has coordinates

- $z_{(i)}$ on $\mathcal{U}_i$, $i = 1, 2$
- $x_1 \neq 0 \Rightarrow [x_1, x_2] = [1, \frac{x_2}{x_1}] = [1, z_{(1)}]$

Fiber has same except fiber transforms by

- $y_{(\ell)} = \left(\frac{x_\ell}{x_i}\right)^k y_{(i)}$
- $y_{(2)} = \left(\frac{x_2}{x_1}\right)^k y_{(1)}$ (and vice versa)
- D_0, D_∞ correspond to $0 / \infty$ of fiber

Now get map

$$\mathbb{C}^2 \setminus \{0\} \longrightarrow M_k \setminus (D_0 \cup D_\infty)$$

$$(x_1, x_2) \mapsto ([x_1, x_2], \; y_{(i)} = x_i^k, \text{ if } x_i \neq 0)$$

This agrees on overlap!! So get a legitimate $k-1$ map $\mathbb{C}^2 \setminus 0 \to M_k - (D_0 \cup D_\infty)$ which somehow „unwraps" the topology.

„It is shown" in [Cal] that the maximal compact subgroup of autos of M_k is $G_k \cong U(n)/\mathbb{Z}_k$, via the natural action on $\mathbb{C}^2 \setminus 0$.

Search for metrics invariant under this action which means $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} u$ for $u(\rho)$ where

$$\rho = \log\left(|x_1|^2 + |x_2|^2 \right)$$

(this is obvious if you close one eye ... G_k sends these points around by the action!)

- u satisfies certain properties to be Kähler:

$$\partial_{\bar{j}} u = u' \cdot \partial_{\bar{j}} \rho$$

$$\partial_i \partial_{\bar{j}} u = u'' \partial_i \rho \, \partial_{\bar{j}} \rho + u' \partial_i \partial_{\bar{j}} \rho$$

- $\partial_{\bar{j}} \rho = \dfrac{x_j}{|x_1|^2 + |x_2|^2}$
- $\partial_i \rho = \dfrac{\overline{x_i}}{|x_1|^2 + |x_2|^2}$
- $\partial_i \partial_{\bar{j}} \rho = \underbrace{\dfrac{\delta_{ij}}{|x_1|^2 + |x_2|^2}}_{A} - \underbrace{\dfrac{\bar{x}_i x_j}{(|x_1|^2 + |x_2|^2)^2}}_{B} = A_{ij} - B_{ij}$

same

- $\partial_i \rho \, \partial_{\bar{j}} \rho = \dfrac{\bar{x}_i x_j}{(|x_1|^2 + |x_2|^2)^2} = B_{ij}$

$$\Rightarrow \partial_i \partial_{\bar{j}} u = u'' B_{ij} + u'(A_{ij} - B_{ij})$$

$$= u' A_{ij} + (u'' - u') B_{ij}$$

So as long as $u'' > 0$, $u' > 0$ this is positive-def!

This is the (local) Kähler condition!

Now want $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} u$ to define a metric inside the class $\alpha = \frac{b}{k}[D_\infty] - \frac{a}{k}[D_0]$. So need for the metric to glue and to have certain assymptotic properties!

$\exists$ smooth $u_0, u_\infty : [0,\infty) \to \mathbb{R}$ w/

(i) $u_0'(0) > 0$, $u_\infty'(0) > 0$

(define metric at infinity)

(ii) $u_0(e^{k\rho}) = u(\rho) - a\rho$

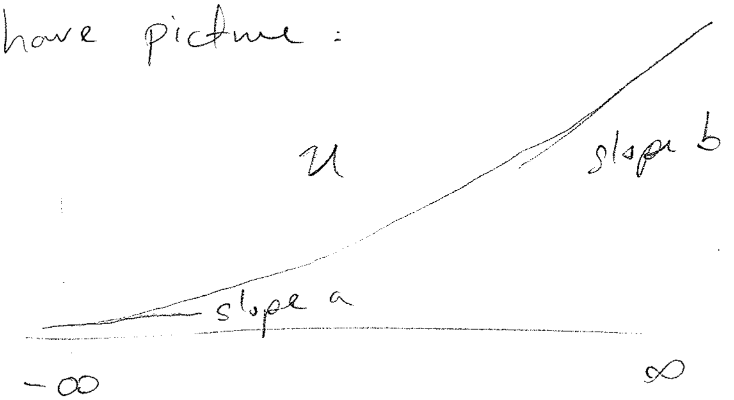
(iii) $u_\infty(e^{-k\rho}) = u(\rho) - b\rho$

$\Big\}$ cohomological restriction ??

Then have

- $u'(\rho) = u_0' \cdot ke^{\rho} + a \underset{\rho\to 0}{\longrightarrow} a$
- $u'(\rho) = u_\infty' \cdot (-k)e^{-k\rho} + b \underset{\rho\to\infty}{\longrightarrow} b$

So have picture:



convex

increasing

Note: Only other possibilities were 0 and $+\infty$ on left/right!

Correspondence:

- $D_0 \longleftrightarrow \rho = -\infty$
- $D_\infty \longleftrightarrow \rho = +\infty$

What is Kähler metric? (From page 8)

- $g_{i\bar{j}} = e^{-\rho} u' \delta_{ij} + e^{-2\rho} \bar{x}_i x_j (u'' - u')$

The determinant of this is:

- $\det g = e^{-2\rho} u' u''$

Now define Ricci potential by:

- $$v = -\log \det g$$
$$= 2\rho - \log u' - \log u''$$

- $$R_{i\bar{j}} = \partial_i \partial_{\bar{j}} v$$
$$= e^{-\rho} v' \delta_{ij} + e^{-2\rho} \bar{x}_i x_j (v'' - v')$$

Now construct reference metric in class α.

- $\hat{u}(\rho) = a\rho + \frac{b-a}{k} \log(e^{k\rho}+1)$

Satisfies magic Calabi conditions trivially.

Now decompose $\hat{\omega}$ into sum of non-neg $(1,1)$-forms.

- $\chi = \pi^* \omega_{FS} \in [e]$ (doesn't see fibers)

$u_\chi(\rho) = \rho$, $\chi = \frac{i}{2\pi} \partial\bar{\partial} u_\chi$

- $u_\theta = 2\log(e^{k\rho}+1)$

$\theta = \frac{i}{2\pi}\partial\bar{\partial} u_\theta$

$\theta \in 2[D_\infty]$

- $$\left.\begin{aligned} \hat{u} &= a u_\chi + \frac{b-a}{2k} u_\theta \\ \hat{\omega} &= a\chi + \frac{b-a}{2k}\theta \end{aligned}\right\}$$

Why the 2? Notational convenience later?

<u>Lemma</u>: $$\left.\begin{aligned} &\cdot\ \chi \wedge \theta > 0 \\ &\cdot\ \int_M \theta^2 > 0 \end{aligned}\right\}$$ <u>Proof later</u>??

$\boxed{\dot{\omega} = -\operatorname{Ric}}$

Theorem 1.1

Let $\omega(t)$ solve Kähler-Ricci w/ ω_0 satisfying the Calabi symmetry.

(a) If $k \geq 2$, flow exists on $[0,T)$ w/ $T = (b_0 - a_0)/2k$ and $(M_k, g(t))$ converges to $(\mathbb{P}^1, a_T g_{FS})$ in G-H as $t \to T$

(b) If $k = 1$, three cases

(i) $b_0 < 3a_0$ same as (a)

(ii) $b_0 = 3a_0$, then $(M_1, g(t)) \to pt$ in G-H as $t \to T = a_0$ (flow exists on $[0,T)$)

(iii) $b_0 > 3a_0$, then $\exists$ on $[0,T)$ w/ $T = a_0$. On compact sets of $M_1 \setminus D_0$, $g(t) \to g_T$ Kähler smoothly. $(\bar{M}, d_T)$ metric completion of $(M_1 \setminus D_0, g_T)$, then $(M_1, g(t)) \longrightarrow (\bar{M}, d_T)$ in G-H; $(\bar{M}, d_T)$ has finite diameter, & $\bar{M} \cong \mathbb{P}^2$ (homeomorphic)

For cases (a) & (b) (i), look at reference metric:

$$\hat{\omega}_t = a_t \chi + \frac{b_t - a_t}{2k} \vartheta$$

$$\xrightarrow[t \to T]{} a_T \chi$$

Define potential $\tilde{\varphi}(t)$ by:

- $\omega(t) = \hat{\omega}_t + \frac{i}{2\pi} \partial\bar{\partial} \tilde{\varphi}(t)$
- $\tilde{\varphi}\big|_{\rho=0} = 0$

<u>Theorem 1.2</u> In case (a) or (b) (i.) let $\omega(t)$ solve K-R w/ Calabi symmetry. Then $\forall\, 0 < \beta < 1$:

(i) $\tilde{\varphi}(t) \longrightarrow 0$ in $\mathcal{C}^{1,\beta}_{\hat{g}_0}(M_k)$ as $t \to T$

(ii) $\forall\, K \subset M_k \setminus (D_\infty \cup D_0)$, $\tilde{\varphi}(t) \to 0$ in $\mathcal{C}^{2,\beta}_{\hat{g}_0}(K)$ as $t \to T$. So on K, $\omega(t) \to a_T \chi$ in $\mathcal{C}^{2,\beta}_{\hat{g}_0}(K)$.

The case $k \geq 2$.

First some estimates that always work!

$$[\omega(t)] \in \alpha_t = \frac{b_t}{k}[D_\infty] - \frac{a_t}{k}[D_0]$$

$$= \frac{b_t - a_t}{k}[D_\infty] - a_t[\ell]$$

T: $\sup\{t \geq 0 \mid \alpha_t \text{ Kähler}\}$

$$\hat{\omega}_t = \underbrace{\alpha_t \chi + \frac{b_t - a_t}{2k} \vartheta}_{\text{Kähler}} \in [\alpha_t]$$

Theorem: $\exists!$ smooth solution of KR flow w/ $\omega_0 \in \alpha_0$ for $t \in [0, T)$. (This is general.)

In our case $T = \frac{b_0 - a_0}{2k}$. So $b_t \to a_t$ but a_t bdd away from 0. Now

$$\hat{\omega}_t = a_t \chi + \frac{b_t - a_t}{2k} \vartheta$$

$$= a_t \chi + (T - t)\,\theta \in \alpha_t$$

From:

- $K_M^{-1} = 2[D_\infty] - (k-2)[\ell]$
- $\theta \in 2[D_\infty]$
- $\chi \in \pi^* \omega_{FS} \in [\ell]$

$$\Rightarrow \quad \theta - (k-2)\chi \in c_1(M)$$

$$\Rightarrow \quad \frac{i}{2\pi} \partial\bar\partial \log \Omega = -\theta + (k-2)\chi,$$

for Ω smooth volume form.

<u>PMAE</u>:

$$\begin{cases} \dot{\varphi} = \log \dfrac{(\hat{\omega}_t + \frac{i}{2\pi}\partial\bar\partial\varphi)^n}{(T-t)\Omega}, \quad \varphi|_{t=0} = \varphi_0, \qquad (*) \\ \hat{\omega}_0 + \frac{i}{2\pi}\partial\bar\partial\varphi_0 = \omega_0 \in \alpha_0 \end{cases}$$

If φ solves $(*)$, then

$$\omega(t) = \hat{\omega}_t + \frac{i}{2\pi}\partial\bar\partial\varphi$$

solves KRF.

Lemma: $\exists\, C$ dep on initial data s.t.

$$|\varphi(t)| \le C, \quad \omega^n(t) \le C\Omega$$

Pf: $\hat{\omega}_t^n = (a_t \chi + (T-t)\theta)^n$

$$\ge n(T-t)\, a_t^{n-1} \chi^{n-1} \wedge \theta$$

$\Rightarrow \exists\, C_1, C_2 > 0$ ind of t: (a_t bdd $>> 0$)

$$C_1 (T-t)\Omega \le \hat{\omega}_t^n \le C_2 (T-t)\Omega$$

~~RD~~ 2nd No!

Now to get bounds for φ:

- $\psi := \varphi - (1 + \log C_2)t$.

Claim: $\sup_{M \times [0,T)} \psi = \sup_M \psi|_{t=0}$.

Else $\exists\, p = (x,t) \in M \times (0,T)$ w/

- $\dot{\psi}(p) \ge 0$
- $\frac{i}{2\pi}\partial\bar{\partial}\psi \le 0$

$$\Rightarrow 0 \le \dot{\psi} \le \log \frac{\hat{\omega}_t^n}{(T-t)\Omega} - 1 - \log C_2 \le -1$$

by $(*)$. Lower bound similar.

Thus ψ and hence φ bdd uniformly.

- For upper bound for ω^n, will bound

$$H = \log \frac{\omega^n}{\Omega} - A\varphi$$

for some A. $\quad \log \omega^n - \underbrace{\log \Omega}$

- $\mathrm{tr}_\omega \omega' = n \dfrac{\omega^{n-1} \wedge \omega'^n}{\omega^n}$ $\quad$ (ω' any 1-1 form)

$$\frac{\partial}{\partial t} H = \underline{\Delta \dot{\varphi}} + \mathrm{tr}_\omega \left(\frac{\partial}{\partial t} \hat{\omega}_t \right) - A\dot{\varphi}$$

$\Delta_{g(t)}$! (since Ω const / by def!)

- $\dot{\varphi} = H + A\varphi - \log(T-t)$ $\qquad \hat{\omega}_t = a_t \chi + (T-t)\theta$
- $\theta \geq 0$

$$\Rightarrow \dot{H} = \Delta H + A\Delta\varphi + (k-2)\mathrm{tr}_\omega \chi$$
$$- \mathrm{tr}_\omega \theta - A\left(H + A\varphi - \log(T-t)\right)$$

$$\leq \Delta H + An - Aa_t \mathrm{tr}_\omega \chi + (k-2)\mathrm{tr}_\omega \chi$$
$$- AH - A^2\varphi + A\log T \quad \text{bdd}$$

A large

$$\leq \Delta H + An - AH - A^2\varphi + A\log T$$

$\Rightarrow$ H bdd above by Max princ., large A.

$$\mathring{H} \le \Delta H + C - AH$$ ☐

Lemma 3.2 : $\exists\, C > 0$ s.t.

$$\operatorname{tr}_\omega \chi = n \frac{\omega^{n-1} \wedge \chi}{\omega^n} \le C$$

Pf: Left to later.

Now apply symmetry!

u det. up to const. so get:

$$\begin{cases} \bullet\ \mathring{u} = \log u'' + \log u' - \rho + c_t \\ \bullet\ c_t = \log u''(0,t) - u'(0,t) \end{cases} \qquad (**)$$

(so $\mathring{u}(0,t) = 0$!)

Also $\rho \mapsto u(\rho, 0)$ initial Kähler so choose $u(0,0) = 0$.

Then $u(0,t) = 0$ on $[0,T)$.

Parabolicity / Existence in general gives solution of (**).

Since metric lies in $\alpha_t = \frac{b_t}{k}[D_\infty] - \frac{a_t}{k}[D_0]$

$$\left.\begin{aligned} \Rightarrow \lim_{\rho \to -\infty} u'(\rho, t) &= a_t \\ \lim_{\rho \to +\infty} u'(\rho, t) &= b_t \end{aligned}\right\} \Rightarrow a_t < u'(\rho,t) < b_t$$

(convexity)

Also deriv give: $\left(u' = \frac{\partial u}{\partial \rho}\ !\right)$

see page 14 of paper for evolution equation of $\dot{u}$!

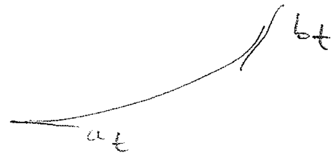
$\boxed{k \geq 2}$

$$t \to T = \frac{b_0 - a_0}{2k}$$

Lemma: (i) $0 < u'(\rho,t) - a_t < 2k(T-t)$

(ii) $\lim_{t \to T} (u(\rho,t) - a_T \rho) = 0$

Pf: (i) is convexity



(iii) Use $u(0,t)=0 \quad \forall t\in[0,T)$, so

$$u(\rho,t) - a_t\rho = \int_0^\rho (u'(s,t)-a_t)\,ds$$

(i) gives $|u(\rho,t) - a_t\rho| \le 2k(T-t)|\rho| \to 0$

as $t\to T$, while $a_t \to a_T$.

Lemma : Along KRF $\omega(t) \ge a_t\chi$.

Pf :

- $g_{i\bar j}(t) = e^{-\rho}u'\delta_{ij} + e^{-2\rho}\bar x_i x_j(u''-u')$
- $\chi_{i\bar j} = e^{-\rho}\delta_{ij} - e^{-2\rho}\bar x_i x_j \qquad (u_\chi = \rho)$

$$\Rightarrow g_{i\bar j}(t) \ge u' e^{-\rho}\left(\delta_{ij} - \frac{\bar x_i x_j}{|x_1|^2+|x_2|^2}\right)$$

$$\ge a_t e^{-\rho}\left(\delta_{ij} - \frac{\bar x_i x_j}{|x_1|^2+|x_2|^2}\right)$$

$$= a_t\chi_{i\bar j}$$

$\square$

<u>Lemma</u>: $\exists C$ dep on initial data s.t.

- $0 < u''(\rho, t) \leq C \min\left(\dfrac{e^{k\rho}}{(1+e^{k\rho})^2}, T-t\right)$

- $|u'''(\rho, t)| \leq C u''(\rho, t)$

$\forall (\rho, t) \in \mathbb{R} \times [0, T)$.

<u>Pf</u>: Uses evolution eq explicitly. Leave out for now.

<u>Theorem 4.1</u>: $\omega(t)$ solves KRF w/ symmetry

(i) $\sup_M \operatorname{tr}_{\hat{g}_0} g \leq C$

(ii) K comp. $\subset M \setminus (D_\infty \cup D_0)$

$$\Rightarrow \sup_K |\nabla_{\hat{g}_0} g|_{\hat{g}_0} \leq C_K$$

<u>Pf</u>: $\operatorname{tr}_{\hat{g}_0} g = \dfrac{u''}{\hat{u}_0''} + \dfrac{u'}{\hat{u}_0'}$ (just computation)

- $\dfrac{u'}{\hat{u}_0'} \leq \dfrac{b_0}{a_0}$

Also: last lemma:

$$\frac{u''}{\hat{u}_0''} = \frac{(1+e^{k\rho})^2}{k(b_0-a_0)e^{k\rho}}\, u'' \le C$$

So get (i). Using (i) need to bound:

$$\frac{\partial}{\partial x_k} g_{i\bar{j}} = e^{-2\rho}(u''-u')(\bar{x}_k \delta_{ij} + \bar{x}_i \delta_{jk}) + e^{-3\rho}\bar{x}_i x_j \bar{x}_k (u''' - 3u'' + 2u').$$

u'', u''' unif bdd

ρ, x_i unif bdd on K done!

$a_0 \le u' \le b_0$ ← top

lower (i) gets $\sqrt{\text{(ii)}}$ upper

Combine this w/ $\boxed{\omega(t) \ge a_t \chi}$ along KRF to get:

<u>Cor</u>: $C^{-1} \le \mathrm{diam}_{g(t)} M \le C$

Theorem 4.2

Define $\tilde{\varphi}(t)$ by

- $\omega(t) = \hat{\omega}_t + \frac{i}{2\pi}\partial\bar{\partial}\tilde{\varphi}$
- $\tilde{\varphi}|_p = 0$

$\forall \beta,\ 0 < \beta < 1$

(i) $\tilde{\varphi} \to 0$ in $\mathcal{C}^{1,\beta}_{\hat{g}_0}(M)$, $t \to T$

(ii) $K \subset M \setminus (D_\infty \cup D_0) \Rightarrow \tilde{\varphi} \to 0$ in $\mathcal{C}^{2,\beta}_{\hat{g}_0}(K)$.

So $\omega(t) \to a_T \chi$ on $\mathcal{C}^{\beta}_{\hat{g}_0}(K)$, $t \to T$.

Pf: Norm of $\tilde{\varphi}$ makes $\tilde{\varphi}(t) = u(t) - \hat{u}_t$.

As $t \to T$, $\hat{u}_t \to a_T u_\chi$. First lemma, $\tilde{\varphi} \to 0$ ptwise.

Taking trace here and using $\mathrm{tr}_{\hat{g}_0} g_t$ bdd see $\Delta_{\hat{g}_0}\varphi$ unif bdd.! A-A $\Rightarrow \hat{\varphi} \to 0$ in $\mathcal{C}^{1,\beta}$.

This gives (i). For (ii) use (ii) of thm to get A-A again.

Theorem: Let $\pi^{-1}(z)$ be fiber of $\pi: M_k \to \mathbb{P}^1$.

Define $\omega_z(t) = \omega(t)|_z$. Then $\forall K \subset M$, $\exists C_K$

$$\sup_{z \in \mathbb{P}^1} \| \omega_z(t) \|_{\mathcal{C}^0(\pi^{-1}(z) \cap K)} \le C_K (T-t)$$

Pf: Fix $K \subset M \setminus (D_0 \cup D_\infty)$. We use

$$\det g = e^{-2\rho} u' u''$$

$\Rightarrow$ ω^2 / Ω unif equiv to u'' $\not\exists$ here use other part!

Lemma
$\Rightarrow$ $\omega^2(x) \le C_K (T-t) \Omega(x)$, $x \in K$

At $x \in K$, choose (z^1, z^2) so

- $\chi = \frac{i}{2\pi} (dz^1 \wedge d\bar{z}^1)$
- $\omega = \frac{i}{2\pi} (\lambda_1 dz^1 \wedge d\bar{z}^1 + \lambda_2 dz^2 \wedge d\bar{z}^2)$
- $\lambda_1, \lambda_2 > 0$

General lemma gives $\mathrm{tr}_\omega \chi \le C$ so

$$\frac{1}{\lambda_1} \le C.$$

But then

$$\lambda_2 \le C' \frac{1}{\lambda_1} \frac{\omega^n(x)}{\Omega} \le C' \cdot C_K (T-t)$$

But this says that $\omega|_{\text{fiber}} = \lambda_2 \, dz_2 \wedge d\bar{z}_2$

so done! ∎

<u>Corollary</u>: $\lim_{t \to T} \left(\sup_{z \in \mathbb{P}^1} \operatorname{diam}_{g(t)} \pi^{-1}(z) \right) = 0$

<u>Pf</u>: Fix $\varepsilon > 0$. By Thm 4.1 (i) $\exists N_\varepsilon$ tub Nbd of $D_0 \cup D_\infty$ s.t. $\forall z \in \mathbb{P}^1$, $t \in [0,T)$,

$$\operatorname{diam}_{g(t)} (\pi^{-1}(z) \cap N_\varepsilon) < \frac{\varepsilon}{2}$$

Also apply last Thm w/ $K = M \setminus N_\varepsilon$ to get for t close to T,

$$\operatorname{diam}_{g(t)} (\pi^{-1}(z) \cap K) < \frac{\varepsilon}{2}.$$

∎

Gromov - Hausdorff Convergence

Def: (X, d_X), (Y, d_Y) compact metric

$d_{GH}((X,d_X),(Y,d_Y))$ is $\inf \varepsilon$ s.t. $\exists\, X \underset{G}{\overset{F}{\rightleftarrows}} Y$

$$|d_X(x_1, x_2) - d_Y(F(x_1), F(x_2))| \le \varepsilon, \quad \forall x_1, x_2 \in X$$

$$d_X(x, G \circ F(x)) < \varepsilon, \quad \forall x \in X$$

(+) the symmetric props in Y.

Theorem: If $k \geq 2$, then $(M, g(t)) \to (\mathbb{P}^1, a_T g_{FS})$ in sense of G-H, as $t \to T$.

Pf: Let $d_t, d_T \leftrightarrow (M, g(t)), (\mathbb{P}^1, a_T \omega_{FS})$.

Let $\varepsilon > 0$. Let $\sigma : \mathbb{P}^1 \to M$ any smooth section not intersecting D_0 or D_∞.

$$\begin{tikzcd}
(X, d_X) \arrow[r, bend left, "F"] \arrow[d, leftrightarrow] & (Y, d_Y) \arrow[l, bend left, "G"] \arrow[d, leftrightarrow] \\
(M, d_t) \arrow[r, bend left, "\pi"] & (\mathbb{P}^1, d_T) \arrow[l, bend left, "\sigma"]
\end{tikzcd}$$

For $x \in M$,

$$d_{g(t)}(x, \sigma \circ \pi(x)) \le \operatorname{diam}_{g(t)}\left(\pi^{-1}(\pi(x))\right) \to 0$$

by Fiber collapse (uniformly!). In other direction $d_{FS}(y, \pi \circ \sigma(y)) = 0$ always.

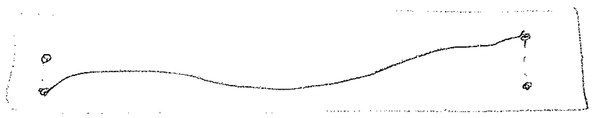
Next prop. Once again choose t close to T so that fibers $\le \frac{\varepsilon}{4}$.

For $x_1, x_2 \in M$, $y_i = \pi(x_i) \in \mathbb{P}^1$. Let γ be geo wrt $a_T g_{FS}$ in $\mathbb{P}^1$ connecting $y_1 \rightsquigarrow y_2$. Then choose tub. nbd N_ε of $D_0 \cup D_\infty$ so that $\tilde{\gamma} = \sigma(\gamma) \subset M \setminus N_\varepsilon$. Then $\tilde{\gamma}(0) = x_1'$, $\tilde{\gamma}(1) = x_2'$ w/ x_i' in same fiber as x_i. On $M \setminus N_\varepsilon$, $g(t)$ conv. to $a_T \pi^* g_{FS}$ $(a_T \chi)$ unif, (theorem 7.2) so if t close to T,

$$L_{g(t)}(\tilde{\gamma}) < L_{a_T g_{FS}}(\gamma) + \varepsilon/2.$$

Then have

$$d_{g(t)}(x_1, x_2) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + d_T(y_1, y_2)$$



So $\quad d_t(x_1, x_2) \le d_T(y_1, y_2) + \varepsilon.$ $\qquad$ Lemma

For other direction use $\quad \omega(t) \ge a_t \chi$

$$d_t(x_1, x_2) \ge \left(\frac{a_t}{a_T}\right)^{1/2} d_T(y_1, y_2)$$

and $\quad a_t \to a_T$ as $t \to T$. So get

$$| d_t(x_1, x_2) - d_T(\pi(x_1), \pi(x_2)) | \le \varepsilon.$$

Other direction, choose $y_1, y_2 \in \mathbb{P}^1$, $x_i = \sigma(y_i)$.
Since $\sigma(\mathbb{P}^1) \cap (D_0 \cup D_\infty) = \phi$,

$$\lim_{t \to T} d_t(x_1, x_2) = d_T(y_1, y_2) \quad \text{unif}$$

by Thm 4.2. $\square$